Exact solution of a stochastic directed sandpile model

Morten Kloster
Department of Physics, Princeton University, Princeton, New Jersey 08544

Sergei Maslov
Department of Physics, Brookhaven National Laboratory, Upton, New York 11973

Chao Tang
NEC Research Institute, 4 Independence Way, Princeton, New Jersey 08540

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We introduce and analytically solve a directed sandpile model with stochastic toppling rules. The model clearly belongs to a different universality class from its counterpart with deterministic toppling rules, previously solved by Dhar and Ramaswamy. The critical exponents are $D_{\|}=7/4,\ \tau=10/7$ in two dimensions and $D_{\|}=3/2,\ \tau=4/3$ in one dimension. The upper critical dimension of the model is three, at which the exponents apart from logarithmic corrections reach their mean-field values $D_{\|}=2,\ \tau=3/2$.

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Numerical and analytical studies of sandpile models of self-organized criticality (SOC) [1] continue to be a subject of considerable research activity. In particular, much effort has recently been invested in establishing the set of universality classes in these systems [2]. The consensus seems to be that the universality class of a d-dimensional sandpile model depends on the following list of questions.

(i) Is it a critical slope or a critical height model? In other words, does a site topple when its local slope or height exceeds a certain threshold value. This concerns the driving mechanism of the model. Critical height models [e.g., Bak-Tang-Wiesenfeld (BTW) model [1]] were studied more extensively in the past and are in general better understood.

(ii) Is sand redistributed isotropically in a toppling event? Accordingly, models can be classified as isotropic or directed (anisotropic). The common knowledge is that this is a relevant parameter, i.e., an arbitrary small anisotropy in toppling rules usually drives the model to the directed universality class.

(iii) Finally, is sand redistributed deterministically or randomly in each individual toppling [3]. In a model with deterministic toppling rules the configuration of the sandpile remains unchanged if every single site on the lattice topples exactly once. This additional symmetry is usually important for the universality class of the model. For example, the deterministic one-dimensional isotropic critical height model (1D BTW) has only trivially distributed avalanches of fractal dimension 2. While the variants with randomness in toppling rules, such as the Zaitsev model [4], Oslo model [5], linear interface model [6], etc., seem to belong to a universality class where avalanches have a noninteger fractal dimension $D \simeq 2.23$ and a probability distribution with a power law exponent $\tau \simeq 1.27$.

Despite many careful numerical and analytical studies of the original BTW sandpile model (which is a deterministic isotropic critical-height model in the above classification), its critical exponents in two dimensions still remain controversial [2]. The situation is somewhat better for directed models. Soon after the original BTW sandpile model [1], Dhar and Ramaswamy introduced and exactly solved in all dimensions its directed counterpart—the Dhar-Ramaswamy (DR) model [7].

Both BTW and DR models have deterministic toppling rules. As far as stochastic models are concerned there is preciously little analytical results. Apart from an exact solution of a model, equivalent to the 1D stochastic directed sandpile [8], stochastic sandpiles were studied only numerically. In this paper we present an analytical study of a stochastic directed sandpile model in all dimensions. Stochastic directed sandpiles were brought to the attention of the community in two recent papers [9], reporting numerical studies of several variants of such models in two dimensions. During the preparation of this paper, there appeared a closely related preprint by Paczuski and Bassler [10] in which an analytical study of the directed stochastic sandpile model was presented and similar results were obtained. In particular, using different analytical arguments they have arrived at the same set of exponents.

The microscopic rules of the stochastic directed sandpile model that we selected to study are closely related to those of the DR model [7]. These rules are modified in the spirit of a stochastic isotropic sandpile model known as the Manna model [11]. It is easier to define our rules in two dimensions, while generalization to higher dimensions is straightforward. A stable configuration of our model is specified by the integer height of the sandpile $z(x_1,x_2)\leqslant 1$ at each point of a 2D square lattice. The lattice has open boundary conditions along the diagonal coordinate, $x_{\|}=x_1+x_2$, and periodic boundary conditions in the transversal direction $x_{\perp}=x_1-x_2$. The sand is added randomly at the line with $x_{\|}=0$ and falls off the edge at $x_{\|}=L_{\|}$. The difference between our model and the DR model lies in toppling rules. In both cases, once the height at any given site exceeds one, this site becomes unstable and loses two grains of sand to its nearest neighbors in the direction of increasing $x_{\|}$. However, while in the DR model each of these two neighbors gets exactly one grain of sand, in our stochastic variant the decision where to move any particular grain is done independently for each grain. In other words, with probability 1/4 both grains end up on the left neighbor, with probability 1/4 they go to the right neighbor, and only with probability 1/2 will each neighbor get one grain as in the DR model. Obviously, on average each neighbor gets one grain, yet the additional stochastic element in the rules drives the model away from the universality class of the DR model [9]. It is easy to see that unlike the deterministic rules of the DR model, the new stochastic rules allow for multiple topplings of some sites within one avalanche. Indeed, let us consider an example where in the first toppling one grain of sand was transferred to each of two sites in the next layer. Let us further assume that these sites both toppled and by chance distributed all four of the resulting grains of sand to the same site in the next layer. This site has received four grains of sand and therefore would topple twice. The numerical simulations [9] confirm the existence of multiple topplings in other variants of a stochastic directed sandpile model.

In order to get an analytical handle on the properties of our model we employ the same trick that was used by one of us to solve a 1D directed stochastic sandpile model [8]. Due to the Abelian [12] nature of the model, we can change the order in which topplings are performed without changing the outcome. It is convenient to do topplings layer by layer. This means that we topple any given site as many times as necessary to make it stable before moving on to the next unstable site, and we topple all unstable sites in one layer (a given $x_{||}$) before toppling any site in the next layer. Let us concentrate on a site with coordinates $x_{||}$ and $x_{\perp}$ immediately after we have finished with topplings in the $(x_{||}-1)$th layer. Assume that two of its neighbors with coordinates $x_{||}-1$ and $x_{\perp}\pm 1$ have toppled, respectively, $n_1=n(x_{||}-1,x_{\perp}-1)$ and $n_2=n(x_{||}-1,x_{\perp}+1)$ times. The average number of grains of sand that our selected site would receive from the previous layer is (n_1+n_2). In the DR model there are no fluctuations around this average. Also, due to the absence of multiple topplings in this deterministic directed model, n_1 and n_2 can be only 0 or 1. Therefore, in the DR model a site can receive either $n_1+n_2=2$ grains, in which case it is guaranteed to topple exactly once, or $n_1+n_2=1$, in which case it can topple with probability 1/2 (i.e., it topples if it had $z=1$ before the transfer and remains stable if it had $z=0$). From this one can show [7] that in the DR model the set of sites which topple at each layer form an interval with no holes inside. The size of this interval as a function of the layer number $x_{||}$ performs an ordinary random walk.

In the stochastic model the relation between the number of topplings in two subsequent layers is more complicated. Let us focus on the behavior of the total number of topplings $N(x_{||})=\Sigma_{x_{\perp}}n(x_{||},x_{\perp})$ in a given layer $x_{||}$. The number of grains of sand transferred from the layer $x_{||}$ to the next layer is simply $2N(x_{||})$. It is easy to see that a site which has received an even number $2k$ of grains of sand from a previous layer will always topple exactly k times and, therefore, will transfer the same $2k$ grains of sand to the layer directly below it. That means that as far as $N(x_{||})$ is concerned, such sites behave in a completely passive manner, i.e. they do not lead to a decrease or an increase of the total number of topplings $N(x_{||})$ from layer to layer. On the other hand, any site which received an odd number $2k+1$ of grains of sand from a previous layer has equal chance to topple k times (if it had $z=0$ before the transfer) or $k+1$ times (if it had $z=1$). In the former case this site would decrease the grain flow $2N(x_{||})$ by one, while in the latter, increase by 1. Let us call any site which has received an odd number of grains from the previous layer an *active site*. The equation, which is the central result of this work, relating the change in the total number of topplings from layer to layer to the number of active sites $N_a(x_{||})$ in a given layer is $N(x_{||})=N(x_{||}-1)+\frac{1}{2}\Sigma_{a=1}^{N_a(x_{||})}\xi_a$, where all ξ_a are -1 or $+1$ with equal probability and independent of each other. These random numbers correspond to whether each of the $N_a(x_{||})$ active sites had the height $z=0$ or $z=1$ before the avalanche started. It is straightforward to demonstrate that, as in the DR model, in the steady state of the directed stochastic model all possible stable configurations of z are equally represented, and, therefore, there are no correlations between the heights at different sites and each height is equally likely to have $z=0$ or $z=1$. It is more convenient to rewrite the above equation in a continuous notation:

$$\frac{dN(x_{||})}{dx_{||}}=\frac{1}{2}\sqrt{N_a(x_{||})}\,\eta(x_{||}), \tag{1}$$

where $\eta(t)$ is a standard Gaussian variable with zero mean and unit variance. This equation describes an unbiased random walk $N(x_{||})$ with a variable step size $\frac{1}{2}\sqrt{N_a(x_{||})}$. A random walk (an avalanche) starts with $N(0)=1$ and ends at $x_{||}$ when $N(x_{||})\leqslant 0$ for the first time. Let us assume that $N(x_{||})$ and $N_a(x_{||})$ in a surviving avalanche scale with $x_{||}$ with the exponents α and α_a, respectively. It follows that $N_a\sim N^{\alpha_a/\alpha}$. We use this relation to eliminate the variable N_a from Eq. (1). The resulting differential equation can be easily solved to give the exponent relations

$$\alpha=\frac{1+\alpha_a}{2}, \quad \tau_{||}=1+\alpha, \tag{2}$$

where $\tau_{||}$ is the exponent of the avalanche length distribution, $p(x_{||})\sim x_{||}^{-\tau_{||}}$. Since by definition the avalanche size (i.e. the total number of topplings) $s=\Sigma_{i=1}^{x_{||}}N(i)\sim x_{||}^{1+\alpha}\sim x_{||}^{D_{||}}$, we recover the well-known exponent relation $\tau_{||}=D_{||}$ for general directed sandpile models.

Note that Eq. (1) applies to the DR model as well as the stochastic directed sandpile models. The difference between these two models lies only in the scaling of the number of active sites with $x_{||}$. As was explained above, in the 2D DR model the only two active sites lie at the edge of the interval of toppled sites. Indeed, only these sites get 1 grain of sand, while the rest get either 0 or 2. Therefore, in the 2D DR model $N_a(x_{||})=2$ is just a constant, $\alpha_a=0$, and the Eq. (1) describes an ordinary random walk in which $N(x_{||})\sim x_{||}^{\alpha}=x_{||}^{1/2}$. The introduction of a stochastic element in particle redistribution dramatically changes the number of active sites at any given layer of the avalanche. Indeed, when grains are redistributed independently, any site which has at least one *toppled neighbor* in the previous layer is equally likely to receive an even or odd number of grains, and therefore, it has a probability 1/2 of becoming active. Thus, in the stochastic model the exponent α_a defines how the number of *distinct sites* that topple at least once, scales with the layer number $x_{||}$. The difference between exponents α and α_a comes solely from the existence of multiple topplings. These two exponents have to obey the inequality $\alpha \geqslant \alpha_a$, and their difference $\alpha - \alpha_a$ determines how the average number of topplings $n_{\mathrm{top}}(x_{||})$ at a given site in the $x_{||}$th layer scales with $x_{||}$: $n_{\mathrm{top}} \sim N/(2N_a) \sim x_{||}^{\alpha-\alpha_a}$.

We proceed with an argument that in the 2D directed stochastic model $\alpha_a = 1/2$, and, therefore, from Eq. (2) $\alpha = 3/4$. It is straightforward to determine the *average* number of topplings $\langle n(x_{||},x_\perp)\rangle$ at a given site $x_{||},x_\perp$, where the average is performed over the whole ensemble of avalanches so that avalanches that die out before reaching this site contribute 0 to the average. As was noted in Ref. [7], due to the conservation of sand and the stationarity of the sandpile profile, $\langle n(x_{||},x_\perp)\rangle$ has to satisfy the diffusion equation with a source:

$$\frac{\partial\langle n(x_{||},x_\perp)\rangle}{\partial x_{||}} = \frac{1}{2}\,\frac{\partial^2\langle n(x_{||},x_\perp)\rangle}{\partial x_\perp^2} + \delta(x_{||})\,\delta(x_\perp). \tag{3}$$

Indeed, *on average* the configuration of the sandpile after an avalanche has to be that before the avalanche minus the extra grain which was added from the source. The above equation is also exact for our stochastic model, where it implies that, like in the DR model, the toppings in the $x_{||}$ layer are spread over the transverse directions as $\Delta x_\perp \sim x_{||}^{1/2}$. In the DR model the toppled sites form a dense interval with no holes, and, therefore, their number is known to scale exactly as $x_{||}^{1/2}$. The situation is somewhat less obvious in the stochastic directed model, where the set of toppled sites can have holes. However, one can argue that these holes would mostly be concentrated near the boundaries of the avalanche in any given layer, while the core of an avalanche will be relatively hole free. Indeed, as will be confirmed later, the 2D stochastic directed model is characterized by multiple topplings, where a site at a layer $x_{||}$ would typically topple $n_{\mathrm{top}}(x_{||}) \sim x_{||}^{1/4}$ times within one avalanche. Since any of the $2n_{\mathrm{top}}$ grains can go to each of the two nearest neighbors independent of others, for large n_{top} the situation where one of the neighbors would receive less than two grains and remain stable is exponentially unlikely. In other words, the creation of a new hole (a region free of topplings) is exponentially suppressed down the slope from the sites, which themselves toppled many times. Therefore, for sufficiently large $x_{||}$ the number of active sites which is proportional to the number of toppled sites, should scale as $N_a \sim x_{||}^{1/2}$. From $\alpha_a = 1/2$ and Eq. (2) one obtains $\alpha = 3/4$ and $\tau_{||} = D_{||} = 7/4$. The exponent for avalanche size distribution is then $\tau = 1 + (\tau_{||} - 1)/D_{||} = 10/7$. These results are in good agreement with both previous numerical simulations of various versions of the stochastic directed sandpile model in two dimensions [9] and our own simulations of the model. In Fig. 1 we present the results of our simulations for the effective exponents $\alpha = d\log N/d\log x_{||}$ and $\alpha_a = d\log N_a/d\log x_{||}$ as a function of $x_{||}$. The numerical exponent α agrees well with the analytical results. The exponent α_a is less clean due to the presence of holes near the boundary of avalanche regions. The exponent first overshoots to a value of almost 0.6 but then clearly goes down so that in the end of the range of our simulations, $x_{||} \sim 100\,000$, it is consistent with our theoretical result $\alpha_a = 1/2$.

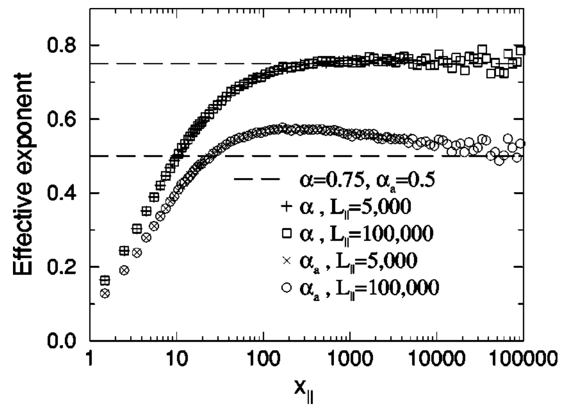

FIG. 1. The scale dependent effective exponents $\alpha = d\log N/d\log x_{||}$ and $\alpha_a = d\log N_a/d\log x_{||}$ as a function of $x_{||}$ for the 2D model, and for two longitudinal system sizes $L_{||} = 5000$ and $L_{||} = 100\,000$.

To address the question of universality Dhar and Ramaswamy [7] have proven that their deterministic directed sandpile model has the same exponents on square, triangular, and partially directed square lattice (in the latter sand is transferred to two nearest neighbors in the same layer and one in the next layer). It is easy to see that our analytical arguments made for the stochastic directed sandpile model defined on the square lattice also apply, with only small modifications, to that defined on the triangular lattice, where in a stable SOC configuration the local height can equally likely be 0, 1, or 2. Our numerical simulations suggest that the model defined on the partially directed square lattice also

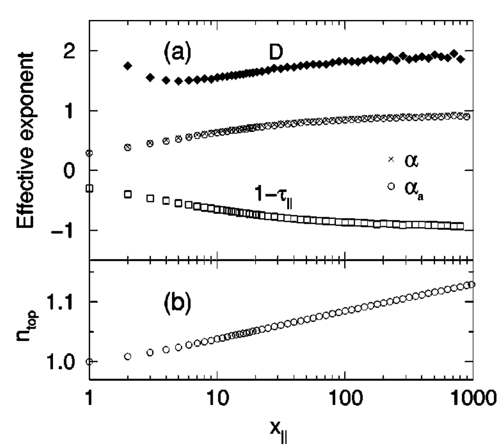

FIG. 2. (a) The effective exponents D, α, α_a, and τ as a function of $x_{||}$ for the 3D model. (b) The expected number of topplings at each site n_{top} as a function of $x_{||}$. Note the logarithmical dependence on $x_{||}$.

has the same exponents. Therefore it is likely that the above critical properties are indeed lattice independent.

Unlike its deterministic counterpart, the stochastic directed sandpile model exhibits a nontrivial scaling even in one dimension. In one variant of its toppling rules, which is essentially identical to the model studied by one of us in Ref. [8], once a height at a given site exceeds one, either one or two grains are transferred to the nearest neighbor down the slope. It is easy to see (for details see Ref. [8]) that this model is equivalent to a 1D random walk so that $N_a = \mathrm{const}$, while the typical number of topplings N scales as a function of $x_{||}=x$ as $N(x) \sim x^{1/2}$. The distribution of avalanche spatial length in this model has an exponent $\tau_{||}=3/2$, while for the avalanche size $\tau=4/3$.

As in the DR model the upper critical dimension for the stochastic directed sandpile model is $d_u=3$. In this dimension the expected number of topplings at each site in a layer $x_{||}$ grows only logarithmically with $x_{||}$. Therefore, $\alpha=\alpha_a$ apart from the logarithmic corrections. From Eq. (2) in this case we get $\alpha=1$, $\tau_{||}=D_{||}=2$, and $\tau=3/2$. This is a standard set of mean-field exponents for any branching (avalanche) process in high enough dimension. In Fig. 2 we plot the numerical effective exponents in the 3D stochastic directed model. They agree well with the mean field values.

In conclusion, we have found an analytic solution of the stochastic directed sandpile model in any dimension. The main difference of this model from its deterministic counterpart—the DR model—lies in the fractal dimension of the set of *active* sites, i.e., sites that can add or remove one grain from the overall flow of sand between two subsequent layers. Whereas in the 2D DR model in any layer there are only two active sites at the edges of the interval of toppled sites, in the 2D stochastic directed sandpile model each of the approximately $x_{||}^{1/2}$ toppled sites in this interval has a 1/2 chance of being active. This leads to an increase in the fractal dimension of an avalanche from $D_{||}=3/2$ to $D_{||}=7/4$ due to multiple topplings. The difference between critical properties of stochastic and deterministic directed models disappears in high dimensions $d \geqslant 3$ where multiple topplings in a stochastic directed sandpile become prohibitively unlikely and all exponents acquire their mean-field values.

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